This is set 1.

1. The *isoelectric point* of glycine is the pH at which it has zero charge. Its charge is $-\frac{1}{3}$ at pH 3.55, while its charge is $\frac{1}{2}$ at pH 9.6. Charge increases linearly with pH. What is the isoelectric point of glycine?

2. The battery life on a computer decreases at a rate proportional to the display brightness. Austin starts off his day with both his battery life and brightness at 100%. Whenever his battery life (expressed as a percentage) reaches a multiple of 25, he also decreases the brightness of his display to that multiple of 25. If left at 100% brightness, the computer runs out of battery in 1 hour. Compute the amount of time, in minutes, it takes for Austin's computer to reach 0% battery using his modified scheme.

3. Compute $\log_2 6 \cdot \log_3 72 - \log_2 9 - \log_3 8$.

This is set 2.

4. Compute the sum of all real solutions to $4^x - 2021 \cdot 2^x + 1024 = 0$.

5. Anthony the ant is at point A of regular tetrahedron $ABCD$ with side length 4. Anthony wishes to crawl on the surface of the tetrahedron to the midpoint of $\overline{BC}$. However, he does not want to touch the interior of face $\triangle ABC$, since it is covered with lava. What is the shortest distance Anthony must travel?

6. Three distinct integers are chosen uniformly at random from the set

$$\{2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030\}.$$

Compute the probability that their arithmetic mean is an integer.

This is set 3.

7. Ditty can bench 80 pounds today. Every week, the amount he benches increases by the largest prime factor of the weight he benched in the previous week. For example, since he started benching 80 pounds, next week he would bench 85 pounds. What is the minimum number of weeks from today it takes for Ditty to bench at least 2021 pounds?

8. Let $\overline{AB}$ be a line segment with length 10. Let P be a point on this segment with $AP = 2$. Let ω_1 and ω_2 be the circles with diameters $\overline{AP}$ and $\overline{PB}$, respectively. Let $\overleftrightarrow{XY}$ be a line externally tangent to ω_1 and ω_2 at distinct points X and Y, respectively. Compute the area of $\triangle XPY$.

9. Druv has a 33×33 grid of unit squares, and he wants to color each unit square with exactly one of three distinct colors such that he uses all three colors and the number of unit squares with each color is the same. However, he realizes that there are *internal sides*, or unit line segments that have exactly one unit square on each side, with these two unit squares having different colors. What is the minimum possible number of such internal sides?

This is set 4.

10. Compute the number of nonempty subsets S of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ such that $\frac{\max S + \min S}{2}$ is an element of S.

11. Compute the sum of all prime numbers p with $p \geq 5$ such that p divides $(p+3)^{p-3} + (p+5)^{p-5}$.

12. Unit square $ABCD$ is drawn on a plane. Point O is drawn outside of $ABCD$ such that lines $\overleftrightarrow{AO}$ and $\overleftrightarrow{BO}$ are perpendicular. Square $FROG$ is drawn with F on $\overline{AB}$ such that $AF = \frac{2}{3}$, R is on $\overline{BO}$, and G is on $\overline{AO}$. Extend segment $\overline{OF}$ past $\overline{AB}$ to intersect side $\overline{CD}$ at E. Compute DE.

This is set 5.

13. How many ways are there to completely fill a 3×3 grid of unit squares with the letters B, M, and T, assigning exactly one of the three letters to each of the squares, such that no 2 adjacent unit squares contain the same letter? Two unit squares are adjacent if they share a side.

14. Given an integer c, the sequence $a_0, a_1, a_2, \ldots$ is generated using the recurrence relation $a_0 = c$ and $a_i = a_{i-1}^i + 2021a_{i-1}$ for all $i \geq 1$. Given that $a_0 = c$, let $f(c)$ be the smallest positive integer n such that $a_n - 1$ is a multiple of 47. Compute

$$\sum_{k=1}^{46} f(k).$$

15. Compute

$$\frac{\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{24}\right)\cos\left(\frac{\pi}{48}\right)\cos\left(\frac{\pi}{96}\right)\cdots}{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{32}\right)\cdots}.$$

This is set 6.

16. Sigfried is singing the ABC's 100 times straight, for some reason. It takes him 20 seconds to sing the ABC's once, and he takes a 5 second break in between songs. Normally, he sings the ABC's without messing up, but he gets fatigued when singing correctly repeatedly. For any song, if he sung the previous three songs without messing up, he has a $\frac{1}{2}$ chance of messing up and taking 30 seconds for the song instead. What is the expected number of minutes it takes for Sigfried to sing the ABC's 100 times? Round your answer to the nearest minute.

17. Triangle $\triangle ABC$ has circumcenter O and orthocenter H. Let D be the foot of the altitude from A to $\overleftrightarrow{BC}$, and suppose $AD = 12$. If $BD = \frac{1}{4}BC$ and $\overleftrightarrow{OH} \parallel \overleftrightarrow{BC}$, compute AB^2.

18. The equation $\sqrt[3]{\sqrt[3]{x-\frac{3}{8}}-\frac{3}{8}} = x^3 + \frac{3}{8}$ has exactly two real *positive* solutions r and s. Compute $r + s$.

This is set 7.

19. Let a be the answer to **Problem 19**, b be the answer to **Problem 20**, and c be the answer to **Problem 21**.

    Compute the real value of a such that

    $$\sqrt{a(101b+1)} - 1 = \sqrt{b(c-1)} + 10\sqrt{(a-c)b}.$$

20. Let a be the answer to **Problem 19**, b be the answer to **Problem 20**, and c be the answer to **Problem 21**.

    For some triangle $\triangle ABC$, let ω and ω_A be the incircle and A-excircle with centers I and I_A, respectively. Suppose $\overleftrightarrow{AC}$ is tangent to ω and ω_A at E and E', respectively, and $\overleftrightarrow{AB}$ is tangent to ω and ω_A at F and F', respectively. Furthermore, let P and Q be the intersections of $\overleftrightarrow{BI}$ with $\overleftrightarrow{EF}$ and $\overleftrightarrow{CI}$ with $\overleftrightarrow{EF}$, respectively, and let P' and Q' be the intersections of $\overleftrightarrow{BI_A}$ with $\overleftrightarrow{E'F'}$ and $\overleftrightarrow{CI_A}$ with $\overleftrightarrow{E'F'}$, respectively. Given that the circumradius of $\triangle ABC$ is a, compute the maximum integer value of BC such that the area $[PQP'Q']$ is less than or equal to 1.

21. Let a be the answer to **Problem 19**, b be the answer to **Problem 20**, and c be the answer to **Problem 21**.

    Let c be a positive integer such that $\gcd(b,c) = 1$. From each ordered pair (x, y) such that x and y are both integers, we draw two lines through that point in the x-y plane, one with slope $\frac{b}{c}$ and one with slope $-\frac{c}{b}$. Given that the number of intersections of these lines in $[0,1)^2$ is a square number, what is the smallest possible value of c? Note that $[0,1)^2$ refers to all points (x, y) such that $0 \le x < 1$ and $0 \le y < 1$.

This is set 8.

22. In $\triangle ABC$, let D and E be points on the angle bisector of $\angle BAC$ such that $\angle ABD = \angle ACE = 90^\circ$. Furthermore, let F be the intersection of $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$, and let O be the circumcenter of $\triangle AFC$. If $\frac{AB}{AC} = \frac{3}{4}$, $AE = 40$, and $\overline{BD}$ bisects $\overline{EF}$, compute the perpendicular distance from A to $\overleftrightarrow{OF}$.

23. Alireza is currently standing at the point $(0, 0)$ in the x-y plane. At any given time, Alireza can move from the point (x, y) to the point $(x + 1, y)$ or the point $(x, y + 1)$. However, he cannot move to any point of the form (x, y) where $y \equiv 2x \pmod 5$. Let p_k be the number of paths Alireza can take starting from the point $(0, 0)$ to the point $(k + 1, 2k + 1)$. Evaluate the sum

$$\sum_{k=1}^{\infty} \frac{p_k}{5^k}.$$

24. Suppose that a, b, c, and p are positive integers such that p is a prime number and

$$a^2 + b^2 + c^2 = ab + bc + ca + 2021p.$$

Compute the least possible value of $\max(a, b, c)$.

This is set 9.

25. For any $p, q \in \mathbb{N}$, we can express $\frac{p}{q}$ as the base 10 decimal $x_1x_2 \dots x_\ell.x_{\ell+1} \dots x_a\overline{y_1y_2 \dots y_b}$, with the digits $y_1, \dots y_b$ repeating. In other words, $\frac{p}{q}$ can be expressed with integer part $x_1x_2 \dots x_\ell$ and decimal part $0.x_{\ell+1} \dots x_a\overline{y_1y_2 \dots y_b}$. Given that $\frac{p}{q} = \frac{(2021)^{2021}}{2021!}$, estimate the minimum value of a. If E is the exact answer to this question and A is your answer, your score is given by $\max\left(0, \left\lfloor 25 - \frac{1}{10}|E - A| \right\rfloor\right)$.

26. Kailey starts with the number 0, and she has a fair coin with sides labeled 1 and 2. She repeatedly flips the coin, and adds the result to her number. She stops when her number is a positive perfect square. What is the expected value of Kailey's number when she stops? If E is your estimate and A is the correct answer, you will receive $\left\lfloor 25e^{-\frac{5|E-A|}{2}} \right\rfloor$ points.

27. Let $S = \{1, 2, 2^2, 2^3, \dots, 2^{2021}\}$. Compute the difference between the number of even digits and the number of odd digits across all numbers in S (written as integers in base 10 with no leading zeros). If E is the exact answer to this question and A is your answer, your score is given by $\max\left(0, \left\lfloor 25 - \frac{1}{2 \cdot 10^8}|E - A|^4 \right\rfloor\right)$.