Time Limit: 60 mins.
Maximum Score: 100 points.
Instructions:

1. All problems require justification unless stated otherwise.
2. You may freely assume results of a previous problem in proving later problems, even if you have not proved the previous result.
3. You may use both sides of the paper and multiple sheets of paper for a problem, but separate problems should be on separate sheets of paper. Label the pages of each problem as 1/2, 2/2, etc., in the upper right hand corner. Write your team ID at the upper-right corner of every page you turn in.
4. Partial credit may be given for partial progress on a problem, provided the progress is sufficiently nontrivial.
5. **Calculators are not allowed!**

Introduction: This power round deals with projective planes, which are geometries without parallel lines. These structures have applications in multiple branches of mathematics: Finite Projective Planes are useful in Algebra and Combinatorics, and the Real and Complex Projective Planes are useful in Geometry and Topology. In this round, we will consider special examples of projective planes and also examine just a few of their properties.

Introduction to Projective Planes (35 Points)

A **projective plane** consists of a set of points P, a set of lines L, and an incidence relation $I \subseteq P \times L$. If $(p, l) \in I$, we say that the point p is incident to the line l (or vice-versa). The incidence relation is required to satisfy the following properties:

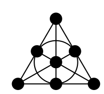
a. Given any two distinct points, there is a unique line incident to both of them. I.e., there is a unique line through any two points.

b. Given any two distinct lines, there is a unique point incident to both of them. I.e., any two lines intersect at a unique point.

c. (Non-degeneracy) There are four points, no three of which are incident to the same line.

From now on, for the sake of brevity, we will often use the notation p_1p_2 to denote the unique line incident to the points p_1 and p_2, and we will use the notation $l \cap l'$ to denote the unique point incident to the lines l and l'. However, remember that these "points" and "lines" are just elements of some set, and they don't necessarily have to correspond to what we usually think of as points and lines. We will now prove some basic properties about projective planes.

1. For our first example, the following image shows a projective plane with 7 points and lines (the circle is also a line, as are the six line segments). This projective plane is known as the Fano Plane. For the following questions about the Fano Plane, no proof is required.



   a. **[2]** For the Fano Plane, how many elements does I, the set encoding the incidence relation, contain? In other words, how many pairs (p, l), consisting of a point and a line in the Fano Plane, are there with p incident to l?

   b. **[2]** Redraw the Fano Plane and circle four points, no three of which are collinear. This will show that the Fano Plane satisfies the non-degeneracy condition.

   c. **[2]** For the previous part, how many such quadruples of points are there in the Fano Plane?

   d. **[2]** Compute the number of permutations f of the 7 points of the Fano Plane such that if p_1, p_2, p_3 are on the same line, so are $f(p_1), f(p_2), f(p_3)$. (Such a permutation is also called a collineation, and collineations will be studied in the final section.)

2. Now, we will use the non-degeneracy condition to prove some general facts about projective planes.

   a. **[2]** Prove that a projective plane must have at least 6 lines.

   b. **[2]** Prove that a projective plane must have at least 5 points.

   c. **[2]** Show that any projective plane contains four lines, no three of which are incident to the same point.

   d. **[3]** Prove that the Fano Plane is the projective plane with the smallest number of points and lines (i.e., prove that no projective plane has fewer than 7 points or fewer than 7 lines, and prove that every projective plane with exactly 7 points and lines is equivalent to the Fano Plane).

Now, note that conditions 1 and 2 for a projective plane are "dual" to each other, since they can be obtained from each other by switching the words "point(s)" and "line(s)." We just proved that the dual of condition 3 always holds as well. Therefore, if any statement is true for all projective planes, its dual statement will automatically be true as well.

3. We will now prove some more facts about finite projective planes (ones with finitely many points and lines).

a. **[3]** Prove that, given any two distinct lines, there is a point not incident to either line.

b. **[2]** Given a line l and a point p not incident to l, construct a bijection between points incident to l and lines incident to p. Use this and the previous result to conclude that, given any two distinct lines, there is a bijection between points incident to the first line and points incident to the second line.

c. **[2]** Assume that a given projective plane has finitely many lines. Prove that if a point p is incident to $n+1$ lines, then every line not incident to p is incident to $n+1$ points.

d. **[3]** Continuing to assume that there are finitely many lines, prove that there's an integer $n \geq 2$ such that every point is incident to $n+1$ lines and every line is incident to $n+1$ points. Conclude that the projective plane has n^2+n+1 points and the same number of lines. Such a finite projective plane will be said to have **order** n.

4. **[8]** A **degenerate projective plane** is a set of points and lines with an incidence relation that satisfies the first two conditions but fails the third, meaning that given any four points, at least three of them are incident to the same line. The empty plane, consisting of no points or lines, vacuously satisfies the first two conditions and fails the third. Prove that a nonempty degenerate projective plane is one of three following types:

   a. There is a line and no points, or there is a point and no lines.

   b. There is a line l incident to all the points. There is a point p such that every other line (if there are other lines) is incident to just the point p.

   c. There is a line l incident to all the points except one (call it p). Every other line (if there are any) is incident to just p and one point that is incident to l.

The Real Projective Plane (25 Points)

In this section, we will consider different but equivalent ways of turning the ordinary plane into a projective plane.

5. The motivation for the definition of a projective plane comes from the ordinary Euclidean plane, $\mathbb{R}^2$. If we define points and lines to be ordinary points and lines in the real plane, and say that point is incident to a line if the point is on the line, then we have a geometry that satisfies conditions 1 and 3. Condition 2 also holds unless the two lines are parallel.

   a. **[3]** Determine the line through the points $(2,3)$ and $(7,-4)$ in the plane. Write the line equation in the form $y = mx + b$. No proof is required.

   b. **[2]** Determine the point of intersection of the lines $y = 12x - 18$ and $y = \frac{1}{2}x + 5$. No proof is required.

   As we noted earlier, $\mathbb{R}^2$ is not quite a projective plane because parallel lines exist. We can remedy this by adding points at infinity and a line at infinity. For every possible slope m (where m can be any real number or even ∞, for vertical lines), adjoin a point p_m so that p_m is incident to every line of slope m. Finally, adjoin a line at infinity that is incident to exactly the points at infinity p_m and no other points. It is clear that this gives a projective plane, which we will call the **extended real plane**.

6. Consider the Euclidean space $\mathbb{R}^3$. Let P be the set of lines through the origin, and let L be the set of planes through the origin. Say that $p \in P$ and $l \in L$ are incident if $p \subseteq l$, i.e., if p is a line in the plane l.

   a. **[4]** Prove that P and L, under the given incidence relation, form a projective plane. We will call this projective plane the **real projective plane**.

b. **[6]** In fact, the real projective plane is equivalent to the extended real plane. To show this, construct a bijection between the points of the extended real plane and lines in $\mathbb{R}^3$ through the origin; and between lines of the extended real plane and planes in $\mathbb{R}^3$ through the origin; in such a way that incidence is preserved.

Hint: in $\mathbb{R}^3$, consider a plane parallel to the xy plane, and use this plane as the base of an extended real plane.

7. Let S^2 be the unit sphere in $\mathbb{R}^3$, let P be the set consisting of pairs of antipodal points, i.e., pairs of the form $\{x, -x\}$, for $x \in S^2$, and let L be the set of great circles on the sphere, which are circles whose center is the center of the sphere. Say that a pair of antipodal points is incident to a great circle if both the points are in the circle.

a. **[5]** Show that this makes the sphere into a projective plane, which we will call the **projective sphere**.

b. **[5]** Show that the projective sphere is equivalent to the real projective plane, i.e., that we can map points and lines in one bijectively to points and lines in the other, in a way that preserves incidence.

Collineations of Projective Planes (40 Points)

A **collineation** of a projective plane (P, L, I) is a function $f : P \cup L \to P \cup L$ such that f maps P bijectively onto P and L bijectively onto L (i.e., f permutes the points and lines), and such that f preserves the incidence relation, meaning that a point p and a line l are incident if and only if $f(p)$ and $f(l)$ are incident. The identity function, mapping all points and lines to themselves, is an obvious collineation. Also, if f, g are collineations, so are $f \circ g$ and f^{-1}.

8. **[8]** Prove that if a collineation fixes all points, it also fixes all lines. By duality, it would follow that any collineation fixing all lines would also fix all points.

9. A collineation f is called a **central** collineation, with center p, if f fixes the point p and fixes all lines incident to p. I.e., given a point $q \neq p$, the line pq is the same as the line $pf(q)$. We say f is **axial**, with axis l, if f fixes the line l and all points incident to l. I.e., if $l' \neq l$ is another line, then l' intersects l at the same point as $f(l')$ intersects l.

a. **[5]** Prove that if a collineation of a projective plane has at least two distinct centers or at least distinct two axes, then it is the identity collineation.

b. **[5]** Prove that an axial collineation that fixes a point p not on the axis is also central, with center p. Similarly, prove that a central collineation that fixes a line l that doesn't contain the center is also axial, with axis l.

10. A collineation with center p and axis l is called a $(p, l)-$collineation.

a. **[6]** Prove that a $(p, l)-$collineation is completely determined by its center, its axis, and by its effect on one point not on the axis and not equal to the center.

b. **[6]** Prove that a $(p, l)-$collineation of the Fano Plane is the identity if p is not incident to l. If p is incident to l, prove that there is always one nonidentity $(p, l)-$collineation.

A projective plane is said to be $(p, l)-$transitive if for any points q_1, q_2 such that neither is the center or on the axis, and such that q_2 is on the same line as p and q_1, there is a $(p, l)-$collineation that sends q_1 to q_2. A projective plane is said to be **complete** if it is $(p, l)-$transitive for any point p and any line l.

11. **[10]** Prove that the real projective plane is complete. It may be helpful to use the equivalence with the extended real plane, the fact that rotations and translations are collineations in the extended real plane, and the fact that rotations fixing the origin are collineations in the real projective plane.