1. Introduction

We begin by introducing some terminology about graphs. A (simple) graph is comprised of a set of vertices $V$ together with a set of edges $E$, which are two-element subsets of $V$. Define the degree of a vertex $v$ as the number of edges in $E$ that contain $v$, and the number of vertices in $G$ to be its order. A bipartite graph is one where the vertices can be split into two sets such that no edge appears between vertices of the same set. Define a cycle to be a set of vertices $v_1, \ldots, v_n$ such that each $v_i$ and $v_{i+1}$ has an edge between them, as does $a_n$ and $a_1$. Define a perfect matching to be a set of edges $E'$ in $G$ where each vertex is the endpoint of exactly one edge in $E'$; for example, in the graph

![Graph with three perfect matchings]

we have the following three perfect matchings:

2. Matchings

1. (a) [1] Draw a perfect matching on the following graph.

![Graph with a perfect matching]

We define $K_n$ to be the complete graph on $n$ vertices; that is, we have $n$ vertices with each pair of distinct vertices being connected by an edge.

(b) [2] Find an expression for the number of perfect matchings of $K_{2n}$, and compute this value for $n = 6$.

(c) [3] Let $P_n$ be a regular $n$ sided polygon with $n$ vertices and $n$ edges. Let $Q_n$ be a graph composed of $n + 1$ copies of $P_n$, called $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_n$, in the following way: Let the vertices of $\mathcal{P}_0$ be $v_1, \ldots, v_n$. Then for $1 \leq k \leq n$, $\mathcal{P}_k$ shares exactly $v_k, v_{k+1}$ with $\mathcal{P}_0$ (taking $v_{n+1} = v_1$), and does not share any edges with any other $\mathcal{P}_i$. An example for $n = 5$ is given below:
Find an expression for the number of perfect matchings of $Q_n$.

(d) [3] Is there a perfect matching of this graph? If so, find one. If not, prove that none exists.

2. (a) [3] What is the number of perfect matchings on the following graph?

(b) [3] What is the number of perfect matchings on this graph?

3. [2] A forest is defined as a graph having no cycles. Show that a forest has at most 1 perfect matching.

4. [5] Show that if $G$ is a graph of order $2n$ so that every vertex of $G$ has degree $\geq n$, then $G$ has a perfect matching.
5. Define $\text{adj}(S)$ as the set of vertices that are adjacent to at least one vertex in $S$. Define $G$ to be a bipartite graph with vertex sets $V_1, V_2$ (that is, all edges have endpoints in a vertex in $V_1$ and $V_2$). Prove that a bipartite graph has a perfect matching if and only if for every subset $S \subseteq V_1$, $|\text{adj}(S)| \geq |S|$ where $|S|$ denotes the size of the set $S$.

3. Recurrences

A recurrence is a sequence $a_n$ where each new term is generated by a function of the ones before it. Often, initial conditions are specified, to give the starting point for the recurrence. For example, a particularly famous recurrence is the Fibonacci sequence, which has initial conditions $F_1 = 1, F_2 = 1$, and recursion formula $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

6. (a) Show that all terms $x_n$ of the recurrence given by $x_n x_{n-2} = x_{n-1}^2 + 1$ with initial conditions $x_0 = 1, x_1 = 1$ are integers.

(b) Find and prove an expression for the $x_n$ in part (a) in terms of the Fibonacci numbers.

We consider sequences $a_n$ given by recurrences of the form

$$a_n a_{n-m} = a_{n-i} a_{n-j} + a_{n-k} a_{n-l},$$

where $m = i + j = k + l$ and with initial conditions $a_0, a_1, \ldots, a_{m-1}$. We call this the three-term Gale-Robinson recurrence. The Somos-4 sequence, $s_n$, a special case of a family of sequences introduced by Michael Somos, is a three-term Gale-Robinson sequence with the following conditions:

$$m = 4, \quad i = 1, \quad j = 3, \quad k = l = 2, \quad s_0 = s_1 = s_2 = s_3 = 1.$$ 

7. Calculate $s_4, s_5, s_6$, and $s_7$.

8. Prove that all terms of the Somos-4 sequence are integers.

9. Suppose instead that we still have $m = 4, i = 1, j = 3, k = l = 2$, but different initial values $a_0, a_1, a_2, a_3$. However, $a_0, \ldots, a_7$ are integers. Let $a_i$ be written in the form $\frac{n_i}{d_i}$, where $n_i, d_i$ are integers and $\gcd(n_i, d_i) = 1$. Show that for any natural number $i$ and any prime $p | d_i$, we have $p | \gcd(a_2, a_3, a_4)$.

4. Tying It All Together

10. Show that the number of matchings $g_n$ for the following graph

satisfies the recurrence $g_n g_{n-2} = g_{n-1}^2 + 2$. 

3
We consider the Aztec Diamond graphs, which consist of a row of 1 square centered atop a row of 3 squares, ..., a row of $2n - 1$ squares, then symmetrically at the bottom. The first few Aztec Diamonds are shown here:

11. [4] Show that every perfect matching of an Aztec Diamond must contain either both the topmost and bottommost edges, or both the leftmost and rightmost edges.

12. (a) [12] The number of perfect matchings on the Aztec Diamond of size $n$ satisfies a Gale-Robinson recurrence. Find the initial conditions, and the $i$, $j$, $k$ and $l$ for this sequence.

   (b) [8] Find an explicit formula for the number of perfect matchings of an Aztec Diamond of size $n$. 