1. Alice is planning a trip from the Bay Area to one of 5 possible destinations (each of which is serviced by only 1 airport) and wants to book two flights, one to her destination and one returning. There are 3 airports within the Bay Area from which she may leave and to which she may return. In how many ways may she plan her flight itinerary?

Solution. Alice has 3 ways of choosing a departing airport, 5 ways of choosing a location from there, and 3 ways to choose a returning airport, giving us $3 \cdot 5 \cdot 3 = 45$ possible flight itineraries.

2. Determine the largest integer $n$ such that $2^n$ divides the decimal representation given by some permutation of the digits 2, 0, 1, and 5. (For example, $2^4$ divides 2150. It may start with 0.)

Solution. The largest integer we can generate with such permutations is 5210, so clearly $n \leq 11$. We see that $2^{10} \cdot 2 = 4096$ and thus $n = 11$ is not possible. However, $2^{10}$ divides 5120, so we have $n = 10$.

3. How many rational solutions are there to $5x^2 + 2y^2 = 1$?

Solution. Note that this is equal to the number of minimal solutions to $5x^2 + 2y^2 = z^2$, where $x, y, z$ are integers. Taking this equation mod 5, we get that $2y^2 = z^2$. But 2 is a quadratic nonresidue modulo 5, so there are no solutions.

4. Determine the greatest integer $N$ such that $N$ is a divisor of $n^{13} - n$ for all integers $n$.

Solution. Using Fermat’s Little Theorem, we see that $n^{13} \equiv n \mod 2, 3, 5, 7, \text{ and } 13$. Thus $2730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ divides $N$. Meanwhile, $2^{13} - 2 = 3 \cdot 2730$. Since $3^{13} - 3$ is not divisible by 9, we must have $N = 2730$.

5. Three balloon vendors each offer two types of balloons – one offers red & blue, one offers blue & yellow, and one offers yellow & red. I like each vendor the same, so I must buy 7 balloons from each. How many different possible triples $(x, y, z)$ are there such that I could buy $x$ blue, $y$ yellow, and $z$ red balloons?

Solution. First, we note that $(x, y, z)$ is an ordered partition of 21. By stars and bars, there are $\binom{23}{2} = 253$ of those. However, since each color is not sold by one of the vendors, we must have $x, y, z \leq 14$. Now, there are $\binom{8}{2} = 28$ partitions where $x \geq 15$. Since no two of $x, y, z$ can simultaneously be $\geq 15$, we have $253 - 3 \cdot 28 = 169$ partitions where no part is greater than 14.

Now, we claim that all of these remaining partitions are indeed possible. Suppose we have $(x, y, z)$ satisfying the conditions. Without loss of generality, assume that $x \leq 7$ (by pigeonhole) and that $y$ is the largest of the three. Then, buy 7 yellow balloons from the blue and yellow vendor, and $x$ blue balloons from the blue and red vendor. Then, buy $7 - x$ red balloons from the blue and red vendor - this is possible because otherwise $z + x \leq 7$, and so $y > 14$ which contradicts our assumption. Finally, we buy the remainder of the balloons from the yellow and red vendor. This gives us a valid way to buy balloons satisfying $(x, y, z)$, and thus we have 169 feasible possibilities of balloon buying.

6. There are 30 cities in the empire of Euleria. Every week, Martingale City runs a very well-known lottery. 900 visitors decide to take a trip around the empire, visiting a different city
each week in some random order. 3 of these cities are inhabited by mathematicians, who will talk to all visitors about the laws of statistics. A visitor with this knowledge has probability 0 of buying a lottery ticket, else they have probability 0.5 of buying one. What is the expected number of visitors who will play the Martingale Lottery?

**Solution.** Consider a random visitor Eve. Eve’s trip is a permutation of the 30 cities, and since she visits them in a random order, each possible permutation is equally likely. Restricting out focus to the ordering of Martingale City and the three mathematician cities, we see that Eve has a \(\frac{1}{4}\) chance of visiting Martingale City before the other three. In this case, she has a \(\frac{1}{2}\) chance of buying a lottery ticket; in the other cases, she has probability 0 of buying one. Thus, Eve has a \(\frac{1}{8}\) chance in total of buying a lottery ticket.

This holds independently for all 900 visitors, giving us an expected value of \(\frac{225}{2}\) lottery tickets sold.

7. At Durant University, an A grade corresponds to raw scores between 90 and 100, and a B grade corresponds to raw scores between 80 and 90. Travis has 3 equally weighted exams in his math class. Given that Travis earned an A on his first exam and a B on his second (but doesn’t know his raw score for either), what is the minimum score he needs to have a 90% chance of getting an A in the class? Note that scores on exams do not necessarily have to be integers.

**Solution.** The probability distribution of Travis’s average score after the first two exams is as follows:

![Graph of probability distribution]

where the area underneath the graph must be 1. Thus, the height of the triangle is \(\frac{1}{5}\). To find the 10th percentile of this graph, we look for a value \(x\) for which the area it encloses is \(\frac{1}{10}\), i.e. \(\frac{1}{10} \cdot \frac{x - 85}{25} = \frac{1}{10}\). This gives us \(x = \sqrt{5}\). Travis will then get an A in the class if \(\frac{2(85 + \sqrt{5}) + n}{3} \geq 90\), or \(170 + 2\sqrt{5} + x \geq 270\). This gives \(x \geq 100 - 2\sqrt{5}\).

8. Two players play a game with a pile with \(N\) coins on a table. On a player’s turn, if there are \(n\) coins, the player can take at most \(\frac{n}{2} + 1\) coins, and must take at least one coin. The player who grabs the last coin wins. For how many values of \(N\) between 1 and 100 (inclusive) does the first player have a winning strategy?

**Solution.** We claim that the second player can only win when \(N = 3(2^k - 1), k \geq 1\). To show this, we use induction. First, we see easily that player 1 wins when \(N = 1\) and \(N = 2\) by grabbing all the coins, but they cannot win when \(N = 3\). Now, suppose we have found the winning strategy for all \(k < N\). Let \(n = 3(2^m - 1)\) be the largest such number that is less than \(N\). Then if \(N - (N/2 + 1) \leq n\), the first player can take away enough coins so that there are \(n\) left on the table, and then they will win. But the statement \(N - (N/2 + 1) \leq n\) is equivalent to \(N/2 - 1 \leq n\), which means \(N \leq 2(3(2^m - 1) + 1) = 2(3 \cdot 2^m - 2) = 3(2^{m+1} - 1) - 1\). Thus the next value of \(N\) for which the second player can win is \(N = 3(2^{m+1} - 1)\), completing the induction. Thus, the first player has a winning strategy for 95 values between 1 and 100.
9. There exists a unique pair of positive integers $k, n$ such that $k$ is divisible by 6, and $\sum_{i=1}^{k} i^2 = n^2$. Find $(k, n)$.

Solution. The sum evaluates to $\frac{k(k + 1)(2k + 1)}{6}$. Since $k, k+1, 2k+1$ are pairwise relatively prime, so are the integers $\frac{k}{6}, k + 1, 2k + 1$, and since their product is a perfect square, each itself is a perfect square. Since $k = (2k + 1) - (k + 1)$, and $k + 1$ and $2k + 1$ have the same parity, $k$ is either odd or divisible by 4. But we are given that $k$ is divisible by 6, so $k$ is not odd and is thus divisible by 4. Since it’s divisible by both 4 and 6, it’s divisible by 12, so say $k = 12m’$. Then $k/6 = 2m’$ is a perfect square, so $m$ is even, and $k$ is in fact divisible by 24. So we have $k = 24m$, and all of $4m, 24m + 1, 48m + 1$ are perfect squares. If we make the obvious choice of $m = 1$, we get 4, 25, 49, all of which are perfect squares, and so $k = 4 \cdot 6 = 24$, and $n^2 = 4 \cdot 25 \cdot 49$, and so $n = 2 \cdot 5 \cdot 7 = 70$. Thus, the answer is $(24, 70)$.

10. A partition of a positive integer $n$ is a summing $n_1 + \ldots + n_k = n$, where $n_1 \geq n_2 \geq \ldots \geq n_k$. Call a partition perfect if every $m \leq n$ can be represented uniquely as a sum of some subset of the $n_i$’s. How many perfect partitions are there of $n = 307$?

Solution. We claim that the number of partitions of $n$ is equivalent to the number of ordered factorizations of $n + 1$. Note that since 1 needs to be represented, $n_k = 1$. Suppose we have $k_1$ 1’s in our partition. Then $k_1 + 1$ must be the next largest number in the partition. Similarly, the next largest must be $(k_1 + 1)(k_2 + 1)$, where we have $k_2$ repetitions of $k_1 + 1$. This continues, until we have $k_i$ repetitions of each $(k_1 + 1) \ldots (k_{i-1} + 1)$ for all $i$ from 1 to $a$ for some $a$. Lastly, we have $k_1 \cdot 1 + k_2(k_1 + 1) + \ldots + k_a(k_1 + 1) \ldots (k_{a-1} + 1) = n$, which simplifies to $(k_1 + 1) \ldots (k_a + 1) = n + 1$. Thus, the number of choices we have for $k_1, \ldots, k_a$ is equal to the number of ordered factorings of $n + 1$. For $n = 307$, we have 308 = $2 \cdot 3 \cdot 53$, which gives us 13 possible ordered factorings.

P1. Find two disjoint sets $N_1$ and $N_2$ with $N_1 \cup N_2 = \mathbb{N}$, so that neither set contains an infinite arithmetic progression.

Solution. One possible answer is to give $N_1 = \{1\} \cup \{4, 5, 6\} \cup \{11, 12, 13, 14, 15\} \cup \ldots$ and $N_2 = \{2, 3\} \cup \{7, 8, 9, 10\} \cup \ldots$.

It is clear here that we cannot have an arithmetic progression in either set, because both of them contain arbitrarily large intervals with no elements of the set in them.

- 2 points for writing a correct distribution into 2 sets
- 4 points for correct justification

P2. Suppose $k > 3$ is a divisor of $2^p + 1$, where $p$ is prime. Prove that $k \geq 2p + 1$.

Solution. WLG assume $k$ is prime. We have $2^p + 1 \equiv 0 \mod k$, and thus $2^p \equiv -1 \mod k$. This means $2^{2p} \equiv 1 \mod k$, and we see that $2p$ is the order of 2 mod $k$. Now, $2^{k-1} \equiv 1 \mod k$ by FLT, so $2p | k - 1$ and so $k \geq 2p + 1$.

- 1 point for looking at the problem modulo $k$
- 1 point for mentioning $2^{k-1} \equiv 1$ by FLT
- 2 points for proving $2p$ is the order of 2 mod $k$
- 2 pts for full solution